

Fuzzy Kernel-Stable Coalitions Between Rational Agents

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ABSTRACT

A large variety of solutions exists for the problem of coalition formation among autonomous agents, at the theoretical level within game theory, and at the practical, algorithmic level, within multi-agent systems. However, one major underlying assumption of algorithmic solutions suggested to date is that the values of the coalitions are known and are certain at the time of coalition formation negotiation. In many practical cases such as in open, dynamically changing environments this assumption does not hold. In this paper we propose an algorithmic solution to the coalition formation problem that overcomes this limitation of previous solutions. Our solution supports fuzzy coalition values and allows agents to form stable coalition configurations. For this, we combine concepts from the theory of fuzzy sets with the game-theoretic stability concept of the Kernel to deduce the new concept of a fuzzy Kernel. We further provide a low-complexity algorithm for forming fuzzy Kernel stable coalitions among agents.

Categories and Subject Descriptors

I.2.11 [Artificial Intelligence]: Distributed Artificial Intelligence—*Coherence and coordination, Multiagent systems*

General Terms

Algorithms, Performance, Economics

Keywords

Fuzzy cooperative games, coalition formation, fuzzy kernel

1. INTRODUCTION

Cooperation within coalitions allows agents to perform tasks that they may otherwise be unable to perform. Co-

operative game theory provides a well developed and mathematically founded theory according to which one can determine which coalitions are beneficial and what coalition configurations are stable and (Pareto) optimal. Game theory itself, however, does not provide algorithms to be used as a coalition formation process. Such algorithms are investigated in the field of multi-agent systems. In recent years, several coalition formation algorithms were proposed, some concentrating on agents that attempt to maximize group utility (e.g., [9]), others addressing self-interested agent that attempt to maximize individual utilities (e.g., [10]). Some solutions tried to reduce complexity [6], compromising optimality, whereas other, exponential solutions sought optimality, proving that without an exponential complexity the solution may be far from optimal [8]. All of these solutions assumed that the values of the coalitions are known for certain before the coalitions are formed and during the formation process. However, in many real-world environments, this assumption does not hold, and values may be known only to a limited degree of certainty. Thus, existing coalition formation methods are inapplicable. In this paper, we address exactly this problem. In particular, we propose a model and an algorithm that allow self-interested agents to form stable coalitions in the face of uncertain values. We do so by fuzzifying concepts of cooperative game theory to allow specification of uncertain coalition values. In our solution, payoff calculation incorporates fuzzy quantities instead of real numbers. Similar work [7] fuzzified the core and the Shapely value. Here, we present a fuzzified version of the Kernel [3], a set-based stability concept which yields stable solutions for every game. The fuzzified Kernel has an intuitively similar interpretation as the crisp Kernel has, however it extends it to contain information about the degree of certainty to which a fuzzy configuration is Kernel-stable. We show that the computational complexity of the fuzzy Kernel is similar to the complexity of the crisp Kernel. We further show that the originally exponential complexity can be reduced to polynomial complexity by placing a cap on the size of coalitions. Finally, we exploit this property to present a polynomial coalition formation algorithm based on the fuzzy kernel.

The paper begins with an introduction to cooperative games and the Kernel (section 2), followed by an introduction to fuzzy quantities in section 3. It proceeds with fuzzifications of the Kernel and related concepts in section 4, and then discusses computational complexity issues in section 5. Sections 6 and 7 present a corresponding fuzzified trans-

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fer scheme for side-payments and the algorithm for forming fuzzy Kernel stable coalitions among agents, respectively. We conclude the paper with an outline of future work in section 8.

2. CRISP KERNEL-STABLE GAMES

In order introduce fuzziness to the cooperative game-theoretic concepts needed for a fuzzified kernel, we briefly remember their original, crisp definitions here.

Definition 1. A cooperative game in characteristic function form is a pair (A, v) with the set of agents A and the *characteristic function* $v : 2^A \mapsto \mathbb{R}$. $v(C)$ is called the value of the coalition $C \subset A$. $v(\emptyset) := 0$.

This value of a coalition C can be viewed as a measure of the payoff achievable by C by cooperating behaviour of its members.

Definition 2. A configuration (C, u) for a game (A, v) specifies a *payoff distribution* $u : A \mapsto \mathbb{R}$ for a *coalition structure* C , a partition of A . $u(a)$, $a \in A$ denotes the *payoff* for agent a . u is called *individually rational* iff $\forall a \in A : u(a) \geq v(a)$ and *efficient* iff $\forall C \in \mathcal{C} : \sum_{a \in C} u(a) = v(C)$

A solution to a game is given by an individually rational and efficient configuration which satisfies a chosen *stability concept*. In this paper we focus on the stability concept of the *kernel*. It is based on the idea that the members of a coalition should be in an equilibrium regarding their "power to object" each others payoff. This power is called *surplus* of an agent over another. It is measured by regarding coalitions not in the coalition structure of the configuration (C, u) in question. The *excess* of such a coalition is the additional payoff its members can gain by actually forming this coalition with respect to their original payoffs according to u .

Definition 3. The *excess* $e(C^*, u)$ of a coalition $C^* \notin \mathcal{C}$ in the configuration (C, u) is given by

$$e(C^*, u) := v(C^*) - \sum_{a \in C^*} u(a)$$

The surplus of an agent a_i over another agent a_k is then defined as the maximum excess of all coalitions including agent a_i but without agent a_k .

Definition 4. The *surplus* s_{ik} of an agent a_i over agent a_k with $a_i, a_k \in C \in \mathcal{C}$, $a_i \neq a_k$, is defined as

$$s_{ik} := \max\{e(C^*, u) \mid C^* \notin \mathcal{C}, a_i \in C^*, a_k \notin C^*\}$$

Please note that this implies the assumption that a_i would be able to gain all of the excess of each considered coalition if it was realized. This might be considered as not very realistic, because the other agents in the coalition would be likely to claim a part of the additional profit for themselves. However, because in the definition of the kernel the surplus is used more like an index but an accurate measurement, this seems to be acceptable.

An agent a_i is said to dominate agent a_k in the configuration (C, u) iff $a_i, a_k \in C \in \mathcal{C}$, $s_{ik} > s_{ki}$ and $u(a_k) > v(a_k)$ hold. The last condition means that agent a_k really has to loose something, i.e. he is better off staying in his current coalition than acting alone. The kernel is then defined as the set of configurations in which no agent dominates another.

Definition 5. The *kernel* K of a game (A, v) is defined as

$$K := \left\{ (C, u) \mid \forall a_i, a_k \in C \in \mathcal{C} : \left[(s_{ik} = s_{ki}) \right. \right. \\ \left. \left. \vee (s_{ik} > s_{ki} \wedge u(a_k) = v(\{a_k\})) \right. \right. \\ \left. \left. \vee (s_{ki} > s_{ik} \wedge u(a_i) = v(\{a_i\})) \right] \right\}$$

Thus, a configuration (C, u) is called a kernel-stable solution of a game (A, v) iff u is individually rational and efficient and (C, u) is an element of the kernel K of (A, v) .

3. FUZZY QUANTITIES

The concept of fuzzy subsets was first introduced by Lotfi A. Zadeh in [13]. It emerged from his idea that it should be possible for elements of a set to belong to a fuzzy subset only to a certain degree. The actual meaning of this degree is application-dependant. For instance, it might be a degree of truth ("truth value") or a possibility in the sense of possibility theory. In the following, we assume some possibilistic interpretation of the fuzziness.

Definition 6. A *fuzzy subset* F of a set M is defined by its *membership function* $\mu_F : M \mapsto [0, 1]$ where $\mu_F(x)$, $x \in M$ is called the *degree of membership* or *membership value* of x in M .

We define some further fuzzy concepts to be used later.

Definition 7.

1. Let F be a fuzzy subset of some set M , and $x \in M$. We write $x \in F$ iff $\mu_F(x) > 0$ and define $\text{support}(F) := \{x \mid x \in F\}$
2. Any fuzzy subset of \mathbb{R} is called a *fuzzy quantity*.
3. Let F be a fuzzy quantity. $\text{size}(F) := \sup\{\text{support}(F)\} - \inf\{\text{support}(F)\}$
4. A fuzzy quantity F is called *normalized* iff $\sup_{x \in \mathbb{R}} \{\mu_F(x)\} = 1$
5. $x \in \mathbb{R}$ with $\mu_F(x) = \max_{y \in \mathbb{R}} (\mu_F(y))$ is called a *modal value* of F .
6. \mathbb{R}^F denotes the set of all fuzzy quantities.
7. r^F denotes a fuzzy quantity with
$$\mu_{r^F}(x) = \begin{cases} 1 & \text{if } x = r \\ 0 & \text{otherwise} \end{cases}, \quad r, x \in \mathbb{R}$$
8. A *fuzzy interval* I is a fuzzy quantity with $\forall x_1, x_2, x_3 \in \mathbb{R}, x_1 < x_2 < x_3 : \mu_I(x_2) \geq \min(\mu_I(x_1), \mu_I(x_3))$
9. Any fuzzy interval N satisfying (a) there exists exactly one $x_m \in \mathbb{R} : \mu_N(x_m) = 1$ and (b) $\exists x_1, x_2 \in \mathbb{R}, x_1 < x_m < x_2 : \mu_N(x) = 0$ for all $x \in \mathbb{R} \setminus [x_1, x_2]$ is called a *fuzzy number*.
10. A *triangular shaped fuzzy number* $(x, y, z)^F$, $x, y, z \in \mathbb{R}$ is a fuzzy number with

$$\mu_{(x,y,z)^F}(r) = \begin{cases} \frac{r-x}{y-x} & \text{if } x < r \leq y \\ \frac{z-r}{z-y} & \text{if } y < r < z \\ 0 & \text{otherwise} \end{cases}, \quad r \in \mathbb{R}$$

For fuzzy quantities and numbers, arithmetic operations can be defined following Zadeh's *extension principle*. For the fuzzy kernel, we need the operations of addition, negation, subtraction and multiplication with a crisp number.

Definition 8. Let $F_1, F_2 \in \mathbb{R}^F$, $x, y, z, a \in \mathbb{R}$.

$$\begin{aligned}\mu_{F_1 \oplus F_2}(x) &:= \sup\{\min(\mu_{F_1}(y), \mu_{F_2}(z)) \mid y + z = x\} \\ \mu_{-F_1}(x) &:= \mu_{F_1}(-x) \\ \mu_{F_1 \ominus F_2}(x) &:= \mu_{F_1 \oplus (-F_2)}(x) \\ \mu_{a \cdot F_1}(x) &:= \begin{cases} \mu_{F_1}(x/a) & \text{if } a \neq 0 \\ \mu_0 & \text{if } a = 0 \end{cases}\end{aligned}$$

Note that for additions and subtractions on F_1 and F_2 with a result F_3 , we have $\text{size}(F_3) = \text{size}(F_1) + \text{size}(F_2)$.

The extension principle is also used to define the fuzzy extension of the max function.

Definition 9. Let $F_1, F_2 \in \mathbb{R}^F$, $x, y, z \in \mathbb{R}$.

$$\mu_{\widetilde{\max}(F_1, F_2)}(x) := \sup\{\min(\mu_{F_1}(y), \mu_{F_2}(z)) \mid \max(y, z) = x\}$$

For convenience, let

$$\begin{aligned}\widetilde{\max}(\{x_1, \dots, x_n\} \subset \mathbb{R}^F) &:= \\ \widetilde{\max}(x_1, \widetilde{\max}(\dots, \widetilde{\max}(x_{n-1}, x_n) \dots))\end{aligned}$$

and $\widetilde{\max}(\{F_1\}) := F_1$

From the definition of the crisp kernel, it is clear that we will need to compare fuzzy quantities to each other in order to determine whether a fuzzy quantity A is greater than a fuzzy quantity B . We cannot use $\widetilde{\max}$ for this comparison because it constructs a new fuzzy quantity out of the membership functions of its operands rather than choosing one of the two to be maximal. Thus, we will need a so-called *ranking method* for fuzzy quantities. Several of such methods have been proposed in the literature (see for example [1]). In section 4, the fuzzy kernel is defined such that configurations are to some degree fuzzy kernel-stable. So we need a ranking operator R that yields a certain degree of a possibilistic measure to a comparison between two fuzzy quantities and thus is a fuzzy subset of $\mathbb{R}^F \times \mathbb{R}^F$.

Definition 10. Let $F_1, F_2 \in \mathbb{R}^F$ and R be a fuzzy subset of $\mathbb{R}^F \times \mathbb{R}^F$. R is called a *fuzzy ranking operator* if $\mu_R(F_1, F_2)$ denotes the degree to which F_1 can be considered "greater" compared to F_2 . R is called a *fuzzy similarity relation* if $\mu_R(F_1, F_2)$ denotes the degree to which F_1 can be considered "similar" to F_2 . Further let G be a fuzzy ranking operator and S a fuzzy similarity relation. We define

$$\begin{aligned}(F_1 \succsim_G F_2) &:= \mu_G(F_1, F_2) \\ (F_1 \approx_S F_2) &:= \mu_S(F_1, F_2)\end{aligned}$$

In the following, we use the "possibility of dominance" PD , which was introduced in [4], as an instance of G . It is defined as

$$F_1, F_2 \in \mathbb{R}^F : (F_1 \succsim_{PD} F_2) := \sup_{x, y \in \mathbb{R}, x \geq y} \{\min(\mu_{F_1}(x), \mu_{F_2}(y))\}$$

As an instance of S , we use the analogously defined similarity relation PS with

$$(F_1 \approx_{PS} F_2) := \sup_{x \in \mathbb{R}} \{\min(\mu_{F_1}(x), \mu_{F_2}(x))\}$$

Further we define a set of maximal elements of a set of fuzzy quantities in the way Mareš did it in [7].

Definition 11. Let X be a set of fuzzy quantities and G a fuzzy ranking operator. The fuzzy subset $X_{\max G}$ of X is given by

$$\forall F_1 \in X : \mu_{X_{\max G}}(F_1) := \min_{F_2 \in X} (F_1 \succsim_G F_2)$$

Thus, $\mu_{X_{\max G}}(F_1)$ denotes the degree to which F_1 can be considered a maximal element of X . For convenience,

$$\max^G X := X_{\max G}.$$

Finally, we will need the logical operations "AND" and "OR" with operands $\in [0, 1]$.

Definition 12. Let $x, y \in [0, 1] : x \wedge y := \min(x, y)$ and $x \vee y := \max(x, y)$

4. FUZZY KERNEL-STABLE GAMES

As we indicated in section 1, negotiation during the coalition forming process might face uncertainties. Such uncertainties could be caused by the possibility of nondeterministic events that hamper the negotiation process and produce incomplete information regarding the values of coalitions. This leads us to the formation of fuzzy-valued coalitions (see also [7]). Other approaches to introduce fuzziness into game theory include the formation of fuzzy coalitions, where agents are members of (multiple) coalitions to a certain degree (see [2]). For fuzzy-valued coalitions, the characteristic function of the game maps to fuzzy values. Thus, the definitions of the basic game-theoretic concepts as given in section 2 have to be revised for the fuzzy-valued case.

Definition 13. A *fuzzy cooperative game* in characteristic function form is a pair (A, v^F) with the set of agents A and the fuzzy characteristic function $v^F : 2^A \mapsto \mathbb{R}^F$. $v^F(C)$ is called the fuzzy value of the coalition C .

In the following we say just "fuzzy game" instead of "fuzzy cooperative game".

Example 1. We give a simple example of a fuzzy game definition with four information agents:

$$A := \{a_1, a_2, a_3, a_4\}$$

Let us suppose these agents have accepted a task to deliver some information to their respective users. The agents' evaluation of the relevance of the available information items to the search tasks is uncertain. The agents now collect as much possibly relevant information as they are able to obtain. An agent can obtain such information either by looking it up in its local database, or by cooperation with another agent. It finally offers the collected information items to its user, who pays some price to get access to those items which are relevant to him/her.

The uncertainty of the relevance of information items to the search task is expressed by the use of fuzzy numbers to summarize the value of a bundle of information items. Thus, let I_d^s denote the fuzzy value of the information items which are locally stored by agent s with respect to the search task of agent d . For example, let's assume agent a_1 has locally stored some information items which he can sell to its own user for "about" 3.5 (monetary units). Further, it is assumed to be certain that a_1 will get no less than 3.0 and no more than 4.0 for them. Thus, we summarize his local information's value for his search task with the triangular fuzzy number $I_{a_1}^{a_1} := (3, 3.5, 4)^F$. The other values of game-relevant bundles of information items are assumed

I_d^s	$s = a_1$	$s = a_2$
$d = a_1$	$(3, 3.5, 4)^F$	$(3, 3.25, 3.5)^F$
$d = a_2$	$(2, 2.25, 2.5)^F$	$(4, 5, 6)^F$
$d = a_3$	$(0.5, 0.65, 2)^F$	$(1.6, 1.75, 1.9)^F$
$d = a_4$	$(5, 5.25, 6)^F$	$(10, 10.5, 11)^F$
\sum^\oplus	$(10.5, 11.65, 14.5)^F$	$(18.6, 20.5, 22.4)^F$
I_d^s	$s = a_3$	$s = a_4$
$d = a_1$	$(0, 0.1, 1)^F$	$(7, 7.25, 8)^F$
$d = a_2$	$(0.4, 0.5, 0.6)^F$	$(5, 6, 7)^F$
$d = a_3$	$(0.5, 0.75, 1)^F$	$(2.5, 2.9, 3.25)^F$
$d = a_4$	$(1, 1.1, 1.25)^F$	$(2, 2.5, 4)^F$
\sum^\oplus	$(1.9, 2.45, 3.85)^F$	$(16.5, 18.65, 22.25)^F$

Table 1: Fuzzy values of information items

to be obtained in a similar way and given in table 1. The last line of this table states the sum of each agent's game-relevant information which gives an easy but very rough indication of an agent's "power" in the upcoming negotiations. For simplicity, we assume that every information is locally available to at most one agent in the game. We restrict size of profitable coalitions to max. two agents by defining $v^F(C_{3,4}) := 0^F$ for all three- and four-agent coalitions $C_{3,4}$. The fuzzy coalition value of a one- or two-agent coalition C is defined as the sum of the fuzzy values of all game-relevant information items available in the coalition: $v^F(C) := \sum_{a_1 \in C}^\oplus \sum_{a_2 \in C}^\oplus I_{a_2}^{a_1}, 1 \leq |C| \leq 2$. The game is summarized as follows:

$$\begin{aligned}
v^F(a_1) &= (3, 3.5, 4)^F & v^F(a_2) &= (4, 5, 6)^F \\
v^F(a_3) &= (0.5, 0.75, 1)^F & v^F(a_4) &= (2, 2.5, 4)^F \\
v^F(a_1, a_2) &= (12, 14, 16)^F & v^F(a_1, a_3) &= (4, 5, 8)^F \\
v^F(a_1, a_4) &= (17, 18.5, 22)^F & v^F(a_2, a_3) &= (6.5, 8, 9.5)^F \\
v^F(a_2, a_4) &= (21, 24, 28)^F & v^F(a_3, a_4) &= (6, 7.25, 9.5)^F \\
v^F(a_1, a_2, a_3) &= 0^F & v^F(a_1, a_2, a_4) &= 0^F \\
v^F(a_1, a_3, a_4) &= 0^F & v^F(a_2, a_3, a_4) &= 0^F \\
v^F(a_1, a_2, a_3, a_4) &= 0^F & &
\end{aligned}$$

With the coalition values being fuzzy, it seems plausible to also fuzzify the payoff distribution: no crisp payoff can be guaranteed as long as it is not known which coalition values will be realized.

Definition 14. A fuzzy configuration is a pair (\mathcal{C}, u^F) with the (crisp) coalition structure \mathcal{C} and the fuzzy payoff distribution $u^F : A \mapsto \mathbb{R}^F$. With a fuzzy coalition values and payoff distributions, the concepts of individual rationality and efficiency also become fuzzy: let G be a fuzzy ranking operator and S a fuzzy similarity relation.

$a \in A : \mu_{\text{indrat}^G}(a) := u^F(a) \succsim_G v^F(a)$ denotes the degree of fuzzy individual G -rationality of agent a .

$\mu_{\text{indrat}^G}(u^F) := \bigwedge_{a \in A} \{\mu_{\text{indrat}^G}(a)\}$ denotes the degree of fuzzy individual G -rationality of u^F .

$\mu_{\text{eff}^S}(u^F) := \bigwedge_{C \in \mathcal{C}} \left\{ \sum_{a_i \in C}^\oplus u^F(a_i) \approx_S v^F(C) \right\}$ denotes the degree of fuzzy S -efficiency of u^F .

Example 2. With $\mathcal{C} := \{\{a_1\}, \{a_3\}, \{a_2, a_4\}\}$, $u^F(a_1) = v^F(\{a_1\})$, $u^F(a_2) = (8.9, 10.4, 12.4)^F$, $u^F(a_3) = v^F(\{a_3\})$ and $u^F(a_4) = (12.1, 13.6, 15.6)^F$, let (\mathcal{C}, u^F) be a fuzzy configuration for the fuzzy game defined in example 1. Here a_2 's and a_4 's payoffs are certainly greater than their respective single-agent coalition values, thus the degree of fuzzy individual PD -rationality is 1.0. The degree of fuzzy PS -efficiency is also 1.0 because $u^F(a_2) \oplus u^F(a_4) = v^F(\{a_2, a_4\})$.

Fuzzy coalition stability concepts define a degree to which given fuzzy configurations are stable. Thus, we define solutions of a fuzzy game as set of fuzzy configurations that

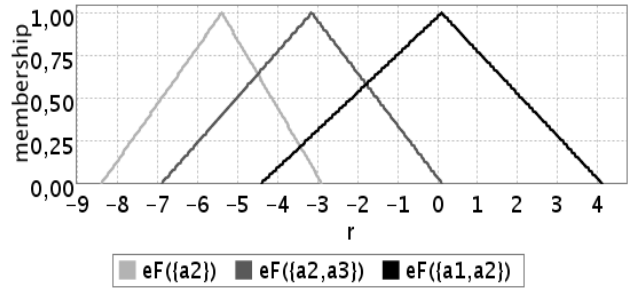


Figure 1: Membership functions of the maximum fuzzy excesses of agent a_2 excluding a_4

satisfy given minimal requirements on the degrees of stability, individual rationality and efficiency.

Definition 15. Let $\mu_{\text{stable}^{SC}}(\mathcal{C}, u^F)$ denote the degree to which a fuzzy configuration (\mathcal{C}, u^F) of a fuzzy game (A, v^F) is stable according to some fuzzy stability concept SC . Let $ir_{\min}, ef_{\min}, st_{\min} \in [0, 1]$, G a fuzzy ranking operator and S a fuzzy similarity relation. The configuration (\mathcal{C}, u^F) is called a $(ir_{\min}, ef_{\min}, st_{\min}, G, S, SC)$ -solution of the fuzzy game (A, v^F) iff $\mu_{\text{indrat}^G}(u^F) \geq ir_{\min}$, $\mu_{\text{eff}^S}(u^F) \geq ef_{\min}$ and $\mu_{\text{stable}^{SC}}(\mathcal{C}, u^F) \geq st_{\min}$.

As we have seen in section 2, the definition of the kernel is based on the concepts of excess and surplus. For the definition of a fuzzy kernel, these concepts also need to be fuzzified. In the case of the excess, this is straight-forward.

Definition 16. The fuzzy excess $e^F(C^*, u^F)$ of a coalition $C^* \notin \mathcal{C}$ in the fuzzy configuration (\mathcal{C}, u^F) is defined as

$$e^F(C^*, u^F) := v^F(C^*) \ominus \sum_{a \in C^*}^\oplus u^F(a)$$

To define the fuzzy surplus, however, we have to keep in mind that the maximum of a set of fuzzy quantities is a fuzzy subset. Thus, the fuzzy surplus should be an element of the (fuzzy) set of maximal fuzzy excesses. However, there could be more than one excess with maximal membership value in this set. Therefore, we will define the surplus as the max of the maximal excesses.

Definition 17. Let E_{ik}^F be the set of fuzzy excesses of an agent a_i excluding agent a_k in the fuzzy configuration (\mathcal{C}, u^F) :

$$E_{ik}^F := \{e^F(C^*, u^F) \mid C^* \notin \mathcal{C}, a_i \in C^*, a_k \notin C^*\}$$

Let G be a fuzzy ranking operator. The fuzzy G -surplus s_{ik}^{FG} in (\mathcal{C}, u^F) of agent a_i over agent a_k is then defined as

$$s_{ik}^{FG} := \widetilde{\max}(\max^G(E_{ik}^F))$$

Example 3. For the fuzzy configuration given in example 2, we consider $\max^{PD}(E_{24}^F)$, the set of maximal fuzzy excesses of agent a_2 excluding agent a_4 , which is illustrated in figure 1. $e^F(\{a_1, a_2, a_3\})$ is not included because its support lies completely to the left of the other three excesses which overlap pairwise. Then, $s_{24}^F = \widetilde{\max}(\max^G(E_{24}^F)) = e^F(\{a_1, a_2\})$.

Now, we are able to give a definition of the fuzzy kernel by just substituting the crisp terms and operators of the definition of the crisp kernel to their fuzzy counterparts.

Definition 18. Let G be a fuzzy ranking operator and S a fuzzy similarity relation. The fuzzy (G, S) -kernel $K^{F^{G,S}}$ of a fuzzy game (A, v^F) is defined by its membership function $\mu_K^{F^{G,S}} : (\mathcal{C}, u^F) \mapsto [0, 1]$ with

$$\mu_{K^{F^{G,S}}}(\mathcal{C}, u^F) := \bigwedge_{a_i, a_k \in \mathcal{C}} \left\{ (s_{ik}^{F^G} \approx_S s_{ki}^{F^G}) \right. \\ \left. \vee (s_{ik}^{F^G} \lesssim_G s_{ki}^{F^G} \wedge u^F(k) \approx_S v^F(\{a_k\})) \right. \\ \left. \vee (s_{ki}^{F^G} \lesssim_G s_{ik}^{F^G} \wedge u^F(i) \approx_S v^F(\{a_i\})) \right\}$$

This definition is perfectly in accordance with the definitions of the crisp kernel; there, the surplus is defined by means of the maximum excess of *possible* coalitions. Agents are further assumed to gain all of the excess, as it was mentioned above. So the crisp excess could be seen as an amount that an agent could *possibly* (but maybe not likely) gain if the corresponding coalition was realized. Thus, already the crisp kernel has some kind of possibilistic interpretation. This is just extended here to the values of the excesses themselves.

Clearly, the actual membership values of configurations in the fuzzy kernel heavily depend on the actual choice for the ranking operator G and similarity relation S . Many have been proposed in the literature and most of them arrive at questionable results in difficult cases, e.g. if non-normalized fuzzy quantities are involved. Thus, choosing the "right" ranking method should be done with respect to a given fuzzy game. Even the possible range of $\mu_{K^{F^{G,S}}}$ depends on that choice and on the membership functions of the coalition values. For example, if we use PD and PS , and some of the coalition values are non-normalized, a configuration (\mathcal{C}, u^F) with $\mu_{K^{F^{PD,PS}}}(\mathcal{C}, u^F) = 1$ might not exist. For practical applications it might thus be necessary to allow only normalized fuzzy quantities.

Example 4. Consider the fuzzy configuration of example 2 again. The most interesting fuzzy comparison in this configuration is $(s_{24}^{F^{PD}} \approx_{PS} s_{42}^{F^{PD}}) \approx 0.84$. It causes the degree of fuzzy (PD, PS) -kernel-stability to be 0.84. As we have seen in example 2, the degrees of fuzzy PS -efficiency and individual PD -rationality are 1.0. Thus, the configuration is a $(1.0, 1.0, 0.84, PD, PS, K^{F^{PD,PS}})$ -solution

For a comparison with the definition of the crisp kernel, consider a game where all coalition values and the values of every payoff function are of the form $r^F, r \in \mathbb{R}$. Then again take $G := PD$, and $S := R$ with for fuzzy quantities $F_1 = r_1^F, F_2 = r_2^F$ let

$$(F_1 \approx_R F_2) := \begin{cases} 1 & \text{if } r_1 = r_2 \\ 0 & \text{otherwise} \end{cases}$$

Then the sets of maximal excesses have exactly one element, with a membership value = 1. Because the result of an addition or subtraction of fuzzy quantities of the form r^F is still of this form, the fuzzy surpluses are also such. Further, all of the comparisons involved to calculate the membership value of a fuzzy configuration in the fuzzy kernel result in degrees of either 0 or 1, resulting in membership values of 0 or 1. Thus, in such a situation, the fuzzy kernel yields, roughly spoken, the same result as the crisp kernel if for every fuzzy quantity r^F in the fuzzy game r is used in the crisp game. If a "real" fuzzy game is considered and the fuzziness is interpreted as possibility, the degree to which a

fuzzy configuration is contained in the fuzzy kernel can be interpreted as the possibility that the configuration is kernel-stable upon realization of the game. "Realization" means that the agents have actually formed coalitions and executed their strategies so that the actual, non-fuzzy coalition values are known in the end.

5. COMPLEXITY

It is clear that in the general case, an actual computation of $\mu_K(\mathcal{C}, u^F)$ needs exponential time with respect to the number of agents in the game because this was already the case with crisp games. However, it is worth to have a closer look on the complexity in order to point out the parts where optimizations and improvements could be made under certain assumptions.

LEMMA 1. For a fuzzy game (A, v^F) , a fuzzy configuration (\mathcal{C}, u^F) and a fuzzy quantity $F \in \mathbb{R}^F$ let

$$n_{agents} = |A|$$

$$n_{fuzzymax} = \max(\max_{C \in 2^A} \{size(v^F(C))\}, \max_{a \in A} \{size(u^F(a))\})$$

The membership value of (\mathcal{C}, u^F) in the kernel K for (A, v^F) can be computed in time

$$Comp_{kernel} = O(2^{2n_{agents}} \cdot n_{agents}^5 \cdot n_{fuzzymax}^2)$$

Sketch of the proof:

The complexity of arithmetic operations and ranking operators on fuzzy quantities F_1 and F_2 can be assumed to be $O(size(F_1) \cdot size(F_2))$ for approximate calculations with general fuzzy quantities.

For the calculation of a fuzzy excess $e(\mathcal{C}, u^F)$, at most n_{agents} fuzzy subtractions and 1 fuzzy addition are required. The maximum size of the fuzzy quantities dealt with during the calculation can reach $(n_{agents} + 1) \cdot n_{fuzzymax}$. The complexity of an operation on fuzzy quantities can thus be bounded by $Comp_{fuzzyop} = O((n_{agents} \cdot n_{fuzzymax})^2)$.

Then, the complexity of a calculation of a fuzzy excess can be summarized with $Comp_{excess} = O(n_{agents} \cdot Comp_{fuzzyop}) = O(n_{agents}^3 \cdot n_{fuzzymax}^2)$.

To compute the fuzzy surplus s_{ik}^F , the fuzzy excesses of an agent will have to be cross-compared. For a set of fuzzy quantities X of size $m = |X|$, m^2 comparisons are made. Since the number of excesses is exponential with respect to n_{agents} , the set of maximal excesses is found in time $O(2^{2n_{agents}} \cdot Comp_{fuzzyop})$. Then, there are maximum $2^{n_{agents}-1}$ max operations, which is covered by $O(2^{2n_{agents}} \cdot Comp_{fuzzyop})$. Thus, the complexity for the calculation of each fuzzy surplus is $Comp_{surplus} = 2^{n_{agents}} \cdot Comp_{excess} + O(2^{2n_{agents}} \cdot Comp_{fuzzyop}) = O(2^{2n_{agents}} \cdot n_{agents}^3 \cdot n_{fuzzymax}^2)$.

For the fuzzy kernel the complexities of \vee and \wedge are $O(1)$. Then, the complexity of a membership value calculation of fuzzy kernel turns out to be $Comp_{kernel} \leq n_{agents}^2 \cdot (2 \cdot Comp_{surplus} + 5 \cdot Comp_{fuzzyop}) = O(2^{2n_{agents}} \cdot n_{agents}^5 \cdot n_{fuzzymax}^2)$.

However, polynomial time can be achieved by limiting the size of the considered coalitions like it was done by Klusch and Shehory (see also [6]). This is especially plausible in situations where communication costs prevent coalitions larger than a certain size to be profitable.

6. TRANSFER SCHEMES

To actually compute stable configurations, iterative techniques have been developed for the crisp case (see also [11]). These are called transfer schemes and specify a sequence of payoff-distributions converging at a stable configuration for a given coalition structure.

For the kernel of a crisp game (A, v) with the configuration (C, u) , an upper bound t_{ki}^{max} for a transfer t_{ki} of agent $a_k \in A$ to agent $a_i \in A$ in one step of the sequence is thereafter given by

$$t_{ki}^{max} := \begin{cases} \min((s_{ik} - s_{ki})/2, u(k) - v(k)) & \text{if } s_{ik} > s_{ki} \\ 0 & \text{otherwise} \end{cases}$$

The transfer is never really paid between the agents. It is only used to compute a stable configuration. In the end, only this stable configuration is part of the solution, and not the sequence of configurations to get there. This is especially important to note when we now come to transfer schemes for fuzzy configurations. A transfer in a fuzzy game is assumed to be a fuzzy quantity, but might be of a crisp form $r^F, r \in \mathbb{R}$. If a fuzzy transfer t^F with $size(t^F) > 0$ is applied to its corresponding configuration, further fuzziness will be introduced into the game, making the transferring algorithm itself fuzzy. Also, a later defuzzification might be affected. We give a transfer scheme for fuzzy games where only fuzzy numbers are allowed, and PD and PS are used as ranking operator and similarity relation. This transfer scheme then yields a an upper bound for $r, r \in \mathbb{R}$ of a transfer which is of the form r^F . In this case, we can ensure the minimum degree of individual rationality of the agent a_k who is the source of the transfer, by examining the rightmost point where $\mu_{u^F(a_k)} = ir_{min}$ and the leftmost point where $\mu_{v^F(\{a_k\})} = ir_{min}$, because if the modal value of $u^F(a_i)$ is less than that of $v^F(a_i)$, $u^F(a_i) \succsim_{PD} v^F(\{a_i\})$ is given by the intersection of the right side of $u^F(a_i)$ with the left side of $v^F(\{a_i\})$. The crisp transfer scheme can then be reformulated to operate on the modal values:

Definition 19. Let \tilde{r} denote a real value r such that for the fuzzy number F , $\mu_F(r)$ is the modal value of F . To find a $(ir_{min}, ef_{min}, st_{min}, PD, PS, K^{F^{PD}, PS})$ -solution for a fuzzy game (A, v^F) where only fuzzy numbers are allowed for v^F and u^F , given the fuzzy configuration (C, u^F) with $\mu_{effs}(u^F) \geq ef_{min}$ and $\mu_{indrat^{PD}}(u^F) \geq ir_{min}$, for a transfer t_{ki} of agent $a_k \in A$ to agent $a_i \in A$ which is of the form $r^F, r \in \mathbb{R}$ an upper bound for r is given by:

$$t_{ki}^{r, max} := \begin{cases} \min((\tilde{s}_{ik}^{F^{PD}} - \tilde{s}_{ki}^{F^{PD}})/2, t_{ki}^{r, max, ir_{min}}) & \text{if } (\tilde{s}_{ik}^{F^{PD}} \succsim_{PD} \tilde{s}_{ki}^{F^{PD}}) > (\tilde{s}_{ki}^{F^{PD}} \succsim_{PD} \tilde{s}_{ik}^{F^{PD}}) \\ 0 & \text{otherwise} \end{cases}$$

with

$$\begin{aligned} u_{Right\ ir_{min}}^F(a_k) &:= \max\{x \mid x \in \mathbb{R}, \mu_{u^F(a_k)}(x) = ir_{min}\} \\ v_{Left\ ir_{min}}^F(a_k) &:= \min\{x \mid x \in \mathbb{R}, \mu_{v^F(\{a_k\})}(x) = ir_{min}\} \\ t_{ki}^{r, max, ir_{min}} &:= u_{Right\ ir_{min}}^F(a_k) - v_{Left\ ir_{min}}^F(a_k) \end{aligned}$$

The definition implies that $ir_{min} \geq st_{min}$, otherwise convergence towards a configuration which holds to both requirements cannot be guaranteed.

The termination criterion for the transfer scheme is given by means of st_{min} .

If $(s_{ik}^F \succsim_{PD} s_{ki}^F) = (s_{ki}^F \succsim_{PD} s_{ik}^F)$, the transfer scheme yields 0 as upper bound for both $t_{ki}^{r, max}$ and $t_{ik}^{r, max}$. But then $(s_{ki}^F \approx_{PS} s_{ik}^F) = 1.0$, and thus the membership of the respective configuration in the kernel is also 1.0.

7. COALITION FORMATION

To show how the previously defined concepts can be used as a basis for coalition formation in games with fuzzy payoffs, we adopt the the Polynomial Kernel-Oriented Coalition Algorithm (KCA) which was introduced in [6], adjusting it to the fuzzy Kernel. Some simplifications are made: only the coalition values are considered in determining the most profitable coalitions and we do not take distributed computation into account. However, it can easily be transformed into a distributed version. We bound the coalition size in order to obtain polynomial execution time. The algorithm is a bilateral coalition formation algorithm, i.e. new coalitions are formed by merging two previously existing ones. In each iteration, a configuration which is a fuzzy solution to the game is formed and contains at most one new coalition. This is because the merging of two coalitions can affect the surpluses, and thus the payoffs, of agents in other coalitions. The configuration is calculated using the transfer scheme given in definition 19 and thus the algorithm is restricted to coalition values which are fuzzy numbers and the use of PD and PS . The algorithm consists of three main parts:

1. *Configuration proposal generation:* each coalition computes possible fuzzy kernel-stable (resp. to st_{min}) configurations for coalition structures in which the active and another coalition are merged. If such a configuration is preferable to the current configuration by the active coalition, it is added to the other coalition's proposal set.
2. (a) *Proposal evaluation:* After the first part is completed, each coalition has a set of proposals from other coalitions and now evaluates which of these proposals are preferable for it, again compared to the current configuration. If a proposal is not preferable, it is deleted from the set.
(b) *Proposal acceptance:* Of the remaining proposals, each coalition accepts one with a maximum gain in benefits in the proposed configuration (by means of PD).
3. *Configuration selection:* Finally, an accepted configuration is chosen to become the next configuration. If there are bilateral accepted coalition structures, i.e. structures in which coalitions C_1 and C_2 are merged and both of them accepted the respective proposals, only their corresponding configuration proposals remain in the set to be chosen from. A configuration with a maximal gain in benefits for the agents in the new coalition is then selected. If there are no accepted proposals, the algorithm terminates.

Definition 20. The Fuzzy Polynomial Kernel-Oriented Coalition Algorithm (KCA-F) is given by the following pseudocode in the context of the fuzzy game (A, v^F) with

- constants $SMin, IMin \geq SMin, EMin$: real; the minimum requirements on the degrees of fuzzy stability, individual rationality and effectiveness, respectively,

for a configuration to be a fuzzy solution. The restriction $IMin \geq SMin$ is required due to the definition of the transfer scheme.

- constant $CSizeMax$: integer; the maximum allowed coalition size (in number of agents).
- operator $PrefROp$: the fuzzy quantity ranking operator used to evaluate the preference of a coalition to merge another. It may be different to that one used for the evaluation of the fuzzy kernel.
- constant $CfgPrefThreshold$: real; the minimum degree of preference of a coalition to merge another coalition.
- function $Size(C)$: integer; returns the number of agents in coalition C .
- function $Pref((C, u^F), (C^*, u^{F*}), C \in \mathcal{C})$: boolean; returns TRUE iff the configuration (C^*, u^{F*}) is preferred by coalition C to the configuration (C, u^F) . That is iff $\min_{a \in C} (u^{F*}(a) \succsim_{PrefROp} u^F(a)) \geq CfgPrefThreshold$
- function $EvalCfg(C, u^F)$: boolean; returns TRUE iff (C, u^F) is a $(IMin, EMin, SMin, PD, PS, K^{FPD, PS})$ -solution where for the evaluation of $K^{FPD, PS}$, only coalitions C with $Size(C) \leq CSizeMax$ are considered.
- function $ComputeCfg((C, u^F), C_1 \in \mathcal{C}, C_2 \in \mathcal{C})$: fuzzy configuration; returns a configuration (C^*, u^{F*}) with $C^* = \{C \mid C = C_1 \cup C_2 \text{ or } C \in \mathcal{C}, C \neq C_1, C \neq C_2\}$. If possible, for the returned configuration

$$EvalCfg(C^*, u^{F*}) = \text{TRUE}$$

holds. The payoff distribution is computed using the transfer scheme given in definition 19.

- function $Gain((C, u^F), (C^*, u^{F*}), C \in \mathcal{C})$: fuzzy number; returns the summarized gain in benefits of the agents in coalition C in the configuration (C^*, u^{F*}) with respect to the configuration (C, u^F) , i.e.

$$\sum_{a \in C} (u^{F*}(a) - u^F(a))$$

- function $Best((C, u^F), CfgGainSet = \{((C_1, u_1^F), Gain_1), \dots, ((C_n, u_n^F), Gain_n)\})$: (fuzzy configuration, fuzzy number); returns a tuple of a configuration and the corresponding gain $((C_i, u_i^F), Gain_i)$ with $((C_i, u_i^F), Gain_i) \in CfgGainSet, 1 \leq i \leq n$ such that with $GSet := \{Gain_1, \dots, Gain_n\}$, $\mu_{\max(GSet)}((Gain_i)) = \max_{g \in GSet} \{\mu_{\max(GSet)}(g)\}$

Further, let $n = |A|$, $a_i \in A, 1 \leq i \leq n$ and u_{init}^F a fuzzy payoff distribution with $u_0^F(a_i) := v^F(a_i), 1 \leq i \leq n$.

```

rnum := 0; C0 := {{a1}, ..., {an}}; u0F := uinitF
repeat
  rnum := rnum + 1; Crnum := Crnum-1; urnumF := urnum-1F
  for all Ci ∈ Crnum do
    PropSetCi := ∅;
  end for
  for all Ci ∈ Crnum do
    for all Ck ≠ Ci ∈ Crnum, Size(Ci ∪ Ck) ≤ CSizeMax do

```

```

      PropCfg := ComputeCfg((Crnum, urnumF), Ci, Ck);
      if EvalCfg(PropCfg) and Pref(Crnum, Prop, Ci) then
        CGain := Gain(Crnum, PropCfg, Ci);
        PropSetCk := PropSetCk ∪ {(PropCfg, CGain)};
      end if
    end for
  end for
  UAProps := ∅; BAProps := ∅;
  for all Ci ∈ Crnum do
    for all (PropCfg, OCGain) ∈ PropSetCi do
      PropSetCi := PropSetCi \ {(PropCfg, OCGain)};
      if Pref(Crnum, PropCfg, Ci) then
        GainSum := OCGain ⊕ Gain(Crnum, PropCfg, Ci);
        PropSetCi := PropSetCi ∪ {(PropCfg, GainSum)};
      end if
    end for
    ((Ca, uaF), Gaina) := Best(Crnum, PropSetCi);
    if ∃((Cpa, upaF), Gainpa) ∈ UAProps : Cpa = Ca then
      UAProps := UAProps \ {(Cpa, upaF), Gainpa};
      BAProps := BAProps ∪ {(Cpa, upaF), Gainpa};
      BAProps := BAProps ∪ {(Ca, uaF), Gaina};
    else
      UAProps := UAProps ∪ {(Ca, uaF), Gaina};
    end if
  end for
  if BAProps ≠ ∅ then
    ((CBA, uBA), GainBA) := Best((Crnum, urnumF), BAProps);
    Crnum := CBA; urnumF := uBA;
  else if UAProps ≠ ∅ then
    ((CUA, uUA), GainUA) := Best((Crnum, urnumF), UAProps);
    Crnum := CUA; urnumF := uUA;
  end if
until Crnum = Crnum-1

```

A remark on complexity: let n_a denote the number of agents. In a configuration, there exist at most n_a coalitions. So for the first part of the KCA-F, the number of computed configurations is thus bounded by n_a^2 . Let $n_{cs} = 2^{CSizeMax}$ ($CSizeMax$ is assumed to be independent from n_a). For the transfer scheme, the complexity for computing a fuzzy surplus was given in section 5, wherein $2^{n_{agents}}$ is to be replaced by n_{cs} : $Comp_{surplus}^{cs} = O(n_{cs} \cdot n_a^3 \cdot n_{fuzzymax}^2)$, with $n_{fuzzymax}$ as in lemma 1. All further calculations in a transfer step are less complex than this. So the complexity of a transfer step is bounded by $Comp_{surplus}^{cs}$. The termination criterion for the transfer scheme is given by means of $SMin$, which, together with PD and PS , plays a similar role as an *allowed error* in the crisp scheme. The crisp theme terminates within $n_a \log(e)$ iteration steps (see [11]), where e is the quotient of the initial and the allowed error of the configuration of the first step. This n_a -independence of the logarithm is maintained in the fuzzy version. Thus $O(n_a)$ transfer steps are made per agent for at most n_a agents. The overall complexity for part one is thus $O(n_a^2 \cdot n_a^2 \cdot Comp_{surplus}^{cs}) = O(n_{cs} \cdot n_a^7 \cdot n_{fuzzymax}^2)$. Similarly, the complexity of $Eval(Cfg)$ is $O(n_{cs} \cdot n_a^5 \cdot n_{fuzzymax}^2)$ (see section 5). All other operations in the algorithm are of less complexity, thus the complexity of the algorithm is given by $O(n_{cs} \cdot n_a^7 \cdot n_{fuzzymax}^2)$.

Example 5. For the game given in example 1 and with $SMin = 0.9$, $IMin = 0.95$, $EMin = 1.0$, $CSizeMax = 4$, $CfgPrefThreshold = 0.9$ and $PrefROp = PD$, the algorithm might proceed as follows (we say "might" because the transfer scheme inherits some nondeterminism due to the tolerances that are given by the choices of $SMin$ and $IMin$). In the first iteration of the repeat loop each agent makes a proposal to every other agent. $\{a_2\}$ and $\{a_4\}$ are chosen to merge because they get the maximum gain in benefits and accept the respective proposals bilaterally. In

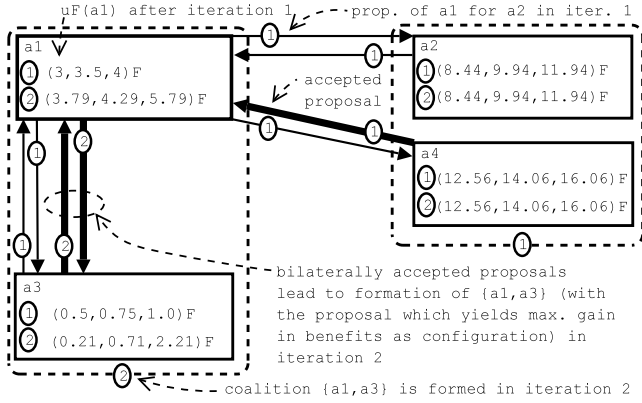


Figure 2: KCA-F coalition formation from the perspective of a_1

the second iteration, $\{a_1\}$ and $\{a_3\}$ propose a merge mutually. They are not able to compute any beneficial proposals for a merge with $\{a_2, a_4\}$ (and vice-versa) because the value of all three- and four-agent coalitions is 0^F . Thus, $\{a_1\}$ and $\{a_3\}$ bilaterally accept their proposals and merge. The complete procedure from the perspective of a_1 together with the resulting configurations is illustrated in figure 2. As we can see there, a_3 "risks" his individual rationality ($\mu_{indrat}^{PD}(a_3) \approx 0.98$) and thus his increase in benefits. This is possible because the respective minimum requirements were set 0.95 and 0.9. This can be interpreted as an optimistic decision, because there still is a possibility that a_3 is better off with this merge than staying alone. All other individual rationalities are 1.0. Further, we have $(s_{13}^{F^{PD}} \approx_{PS} s_{31}^{F^{PD}}) \approx 0.98$ and $(s_{24}^{F^{PD}} \approx_{PS} s_{42}^{F^{PD}}) \approx 0.96$. Thus, the final configuration (in the third iteration, no proposals at all are made, and the algorithm terminates) is a $(0.98, 1.0, 0.96, PD, PS, K^{F^{PD}, PS})$ -solution. The payoffs resemble the agents' abilities quite intuitively. Although a_4 has slightly less game-relevant information available than a_2 , a_4 's threat to coalize with a_1 instead is considerably larger than a_2 's. a_1 and a_3 clearly have much less game-relevant information, thus their coalition value and consequently their payoffs are much smaller. However, both a_1 's game-relevant information and coalitional possibilities are much more valuable than those of a_3 , which explains the payoff-distribution in favor of a_1 very well.

8. CONCLUSION

In the setting of fuzzy-valued cooperative game theory, we fuzzified some game-theoretic concepts such as configurations, individual rationality and efficiency to introduce the concept of fuzzy kernel-stable coalitions. With these definitions, it is possible to specify cooperative games involving uncertain information and to find good candidates of fuzzy configurations to be a solution for the game by evaluating their membership value in the fuzzy kernel. It has been pointed out that the choice of a fuzzy ranking method is an essential aspect of this procedure. A transfer scheme to calculate fuzzy kernel-stable configurations, and a coalition formation algorithm, the KCA-F, have been provided. The procedure of coalition formation of the KCA-F was illus-

trated with the help of an explanatory example. The complexities of an evaluation of a configuration's membership value, the calculation of fuzzy kernel-stable configurations and the KCA-F as a whole have been shown to be exponential, but could be reduced to polynomial time by limiting the size of considered coalitions. Because a precondition of our approach was that all agents in a game share the same understanding about the fuzziness of the data, further work includes the identification of suitable methods to compute the game-wide accepted membership functions from agent-specific beliefs. Also, proper defuzzification methods for the fuzzy payoff-distribution, which will be needed once the coalitions performed their tasks and got their actual (crisp) payoffs, will have to be found. To sum it up more generally, exact specifications of environments for applications of fuzzy valued cooperative games need to be developed.

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